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Arithmetic Meyer sets and finite automata

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Abstract

Non-standard number representation has proved to be useful in the speed-up of some algorithms, and in the modelization of solids called quasicrystals. Using tools from automata theory, we study the set $\mathbb{Z}_\beta$ of β-integers, that is, the set of real numbers which have a zero fractional part when expanded in a real base β, for a given $\beta > 1$. In particular, when β is a Pisot number—like the golden mean—, the set $\mathbb{Z}_\beta$ is a Meyer set, which implies that there exists a finite set F (which depends only on β) such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Such a finite set F, even of minimal size, is not uniquely determined. In this paper, we give a method to construct the sets F and an algorithm, whose complexity is exponential in time and space, to minimize their size. We also give a finite transducer that performs the decomposition of the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ as a sum belonging to $\mathbb{Z}_\beta + F$.

1. Introduction

It is well known that the choice of an adequate number representation can speed-up some algorithms. For instance, the signed-digit number representation consists of an integer base $\beta > 1$ and a set of signed digits $\{-a, -a+1, \ldots, a\}$ with $\beta/2 \leqslant a \leqslant \beta - 1$; in such a system a number may

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have several representations. This property of redundancy allows fast addition and multiplication, and also to design on-line algorithms, see [3,8,10]. A complex base like $-1+i$ allows to expand any complex number as a sequence of digits 0 and 1 with no splitting of the real and the imaginary part, and is convenient for some algorithms, see [26].

Special attention has been raised to the case where the base β is a non-integer real number. In this case, the number system is naturally redundant, see [25]. The well-known fact that addition is computable by a finite transducer when the base is an integer can be extended to some special type of non-integer base. A *Pisot number* (or a Pisot–Vijayaraghavan number) is an algebraic integer >1 such that all its algebraic conjugates have modulus strictly less than one. The natural integers and the golden mean are Pisot numbers. It happens that, when the base is a Pisot number, addition is computable by a finite transducer as well [11]. So Pisot numbers can be considered as a nice generalization of the natural integers.

Another domain where these numbers play an important role is the modelization of the so-called "quasicrystals." The classical crystallography prescribes entirely the possible orders of symmetry of crystals: it can be 2, 3, 4 or 6. When physicists observed in the eighties new alloys presenting a symmetry of order 5, and a long-range aperiodic order, the mathematical notion of quasicrystals had already been introduced by Meyer [20–24] in order to define a generalization of ideal crystalline structures. So the name of Meyer set was given to a mathematical idealization of these solids.

A set X of $\mathbb{R}^d$ is a *Meyer set* if it is a *Delaunay set*—that is, a set which is uniformly discrete and relatively dense—and if there exists a finite set F such that the set of differences $X-X$ is a subset of $X+F$. Meyer [20] has shown that if X is a Meyer set and if $\beta>1$ is a real number such that $\beta X \subset X$ then β must be a Pisot or a Salem number.[1] Conversely, for each d and for each Pisot or Salem number β, there exists a Meyer set $X \subset \mathbb{R}^d$ such that $\beta X \subset X$. Note that all the quasicrystals observed in the real world are linked to quadratic Pisot numbers, namely $\frac{1+\sqrt{5}}{2}$, $1+\sqrt{2}$ and $2+\sqrt{3}$, see [4].

In classical crystallography, crystals are sitting in a lattice, whose vertices are indexed by integers. In quasicrystallography, the points of a quasicrystal are labelled by the so-called β-integers, which are real numbers such that their fractional part is equal to 0 when they are expanded in base β (see Section 2 for definitions). So numeration in real base β is an adequate tool for the description of these solids. As a consequence, β-integers are handled as words, and the set of the expansions of β-integers is known to be recognizable by a finite state automaton when β is a Pisot number (see [13]).

When β is a Pisot number, the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set, see [7]. In this paper, by means of automata theory tools, we give an algorithm that computes a minimal set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

With a geometrical approach, Lagarias [17,18] has given a general construction of a set F satisfying $X-X \subset X+F$ for any Meyer set X. But the sets obtained are huge and no method of minimization of these sets is known. Minimal sets F are given in [7] for $\mathbb{Z}_\beta$ when β is a quadratic Pisot unit. When β is a quadratic Pisot number, a possible set F for $\mathbb{Z}_\beta$ is exhibited in [14].

[1] A *Salem number* is an algebraic integer such that every conjugate has modulus smaller than or equal to 1, and at least one of them has modulus 1.

The method consists in giving a bound on the length of the fractional part of the β-expansion of the sum (respectively the difference) of two β-expansions.

In this work, we use different methods, coming from automata theory. We first give the minimal finite automata describing the formal addition and subtraction, that is the digit-sum and digit-difference, of β-integers in the case where β is a Parry number (see definition in Section 2). Every Pisot number is a Parry number, but the converse does not hold.

We then give a construction of a finite set F of minimal size such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ making use of automata. This algorithm of minimization, which is the first known, is exponential in time and space. It also computes a finite transducer that performs the decomposition of the result of the formal subtraction $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ into a sum belonging to $\mathbb{Z}_\beta + F$.

A preliminary version of this work has been presented in [2].

2. Preliminaries

Let A be a finite alphabet. A concatenation of letters of A is called a *word*. The set A^* of all finite words equipped with the operation of concatenation and the empty word ε is a free monoid. We denote by a^k the word obtained by concatenating k letters a. The length of a word $w = w_0w_1 \cdots w_{n-1}$ is denoted by $|w| = n$. One considers also infinite words $v = v_0v_1v_2 \cdots$ The set of infinite words on A is denoted by $A^{\mathbb{N}}$. An infinite word v is said to be *eventually periodic* if it is of the form $v = wz^\omega$, where w and z are in A^* and $z^\omega = zzz \cdots$ A *factor* of a finite or infinite word w is a finite word v such that $w = uvz$; if $u = \varepsilon$, the word v is a *prefix* of w.

The *lexicographic order* for infinite words over an ordered alphabet is defined by $v <_{\text{lex}} w$ if there exist factorizations $v = uav'$ and $y = ubw'$, for some word $u \in A^*$, $a, b \in A$ such that $a < b$, and $v', w' \in A^{\mathbb{N}}$.

2.1. β-Expansions

Definitions and results can be found in [19, Chapter 7]. Let $\beta > 1$ be a real number. Any non-negative real number x can be represented in base β by the following greedy algorithm [27].

Denote by $\lfloor . \rfloor$ and by $\{.\}$ the integral part and the fractional part of a number. There exists $k \in \mathbb{Z}$ such that $\beta^k \leqslant x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. For $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$. Then $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$ If $x < 1$, we get $k < 0$ and we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. The sequence $(x_i)_{k \geqslant i \geqslant -\infty}$ is called the (greedy) *β-expansion* of x, and is denoted by

$$\langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots$$

most significant digit first. The part $x_{-1}x_{-2} \cdots$ after the “decimal” point is called the *β-fractional part* of x.

The digits x_i are elements of the *canonical* alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$ and $A_\beta = \{0, \ldots, \beta - 1\}$ otherwise. When a β-expansion ends in infinitely many zeroes, it is said to be *finite*, and the 0’s are omitted.

A finite or infinite word w on A_β which is the β-expansion of some non-negative number x is said to be *admissible*. Leading 0's are allowed. The *normalization* on an alphabet of digits $D \supseteq A_\beta$ is the function that maps a word $w = w_k \cdots w_0$ on D onto the β-expansion of its numerical value $\sum_{i=0}^{k} d_i \beta^i$ in base β. The same notion exists for infinite words. Addition is a particular case of normalization: first add digitwise two β-expansions; this gives a word on the alphabet $\{0, \ldots, 2\lfloor\beta\rfloor\}$; then normalize to obtain the result. It is known that for every alphabet D normalization is computable by a finite transducer [11].

Denote by D_β the set of β-expansions of numbers of $[0, 1)$ and by σ the shift defined by $\sigma(x_k x_{k-1} \cdots) = x_{k-1} x_{k-2} \cdots$ Then D_β is shift-invariant. Let S_β be its closure in $A_\beta^{\mathbb{N}}$. The set S_β is a symbolic dynamical system, called the *β-shift*. There is a peculiar representation of the number 1 which can be used to characterize the elements of the β-shift. It is denoted by $d_\beta(1)$, and computed by the following process [27]. Let the *β-transform* be defined on $[0, 1]$ by $T_\beta(x) = \beta x \bmod 1$. Then $d_\beta(1) = (t_i)_{i \geqslant 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$. Note that $\lfloor\beta\rfloor = t_1$.

Set $d_\beta^*(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$ if $d_\beta(1) = t_1 \cdots t_m$ is finite, and $d_\beta^*(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite. Then a sequence s of natural integers is an element of D_β if and only if for every $p \geqslant 1$, $\sigma^p(s)$ is strictly less in the lexicographic order than $d_\beta^*(1)$, see Parry [25].

The numbers β such that $d_\beta(1)$ is eventually periodic are called *Parry numbers*, and *simple* Parry numbers in the case where $d_\beta(1)$ is finite. When β is a Pisot number then $d_\beta(1)$ is finite or infinite eventually periodic [5,29].

Example 1. If $\beta = \frac{1+\sqrt{5}}{2}$, then $d_\beta(1) = 11$ and $d_\beta^*(1) = (10)^\omega$.

If $\beta = \frac{3+\sqrt{5}}{2}$, then $d_\beta(1) = 21^\omega = d_\beta^*(1)$.

The set $\mathbb{Z}_\beta$ of *β-integers* is the set of real numbers x such that the β-fractional part of $|x|$ is equal to 0,

$$\mathbb{Z}_\beta = \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} = \mathbb{Z}_\beta^+ \cup \mathbb{Z}_\beta^-,$$

where $\mathbb{Z}_\beta^+$ is the set of non-negative β-integers, and $\mathbb{Z}_\beta^- = -\mathbb{Z}_\beta^+$. Observe that

$$-\mathbb{Z}_\beta = \mathbb{Z}_\beta \quad \text{and} \quad \beta(\mathbb{Z}_\beta) \subset \mathbb{Z}_\beta.$$

Notice that, if β is an integer, the set of β-integers is just $\mathbb{Z}$.

Denote L_β^+ the set of β-expansions of the elements of $\mathbb{Z}_\beta^+$ with possible leading 0's; then L_β^+ is equal to the set of finite factors of S_β.

2.2. Meyer sets

We recall here several definitions and results from Meyer that can be found in [20–24]. A set $X \subset \mathbb{R}^d$ is *uniformly discrete* if there exists a positive real r such that for any $x \in \mathbb{R}^d$, the open ball of center x and radius r contains at most one point of X. If $Y \subset X$ and X is uniformly discrete, then Y is uniformly discrete. A set $X \subset \mathbb{R}^d$ is *relatively dense* if there exists a positive real R such that for any $x \in \mathbb{R}^d$, the open ball of center x and radius R contains at least one point of X. If $X \subset Y$ and X

is relatively dense, then Y is relatively dense. A set X is a *Delaunay set* if it is uniformly discrete and relatively dense.

The set $X - X$ is the set $\{x - y \mid x \in X,\ y \in X\}$. A set X is a *Meyer set* if it is a Delaunay set and there exists a finite set F such that $X - X \subset X + F$. Lagarias [18] has proved that a set X is a Meyer set if and only if both X and $X - X$ are Delaunay sets. Note that when X is a Delaunay set, then $X - X$ is relatively dense, but not necessarily uniformly discrete. For example, $X = \{n + \frac{1}{|n|+2} \mid n \in \mathbb{Z}\}$ is a Delaunay set and $X - X$ has 1 as point of accumulation.

Lemma 1. *For β a real number > 1, the set $\mathbb{Z}_\beta$ of β-integers is relatively dense.*

Proof. Indeed any non-negative real number x can be expanded as

$$\langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots$$

thus $x = z + r$ with $z = \sum_{i=0}^{k} x_i \beta^i \in \mathbb{Z}_\beta^+$, and $0 \leqslant r = \sum_{i<0} x_i \beta^i < 1$ is the β-fractional part of x. Thus, the maximal distance between two consecutive elements of $\mathbb{Z}_\beta$ is equal to 1. $\square$

The following result is already proved in [7], but we give here a different proof.

Proposition 1. *If β is a Pisot number, then the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set.*

Proof. Let us prove that $\mathbb{Z}_\beta$ is uniformly discrete when β is a Pisot number. Indeed, the minimal distance between two consecutive points a and b of $\mathbb{Z}_\beta$ with $\langle |a| \rangle_\beta = a_N \cdots a_0$ and $\langle |b| \rangle_\beta = b_N \cdots b_0$ is equal to the minimum of $|\sum_{i=0}^{N} (a_i - b_i)\beta^i|$.

Since an integral linear combination of algebraic integers is still an algebraic integer, $\sum_{i=0}^{N} (a_i - b_i)\beta^i$ is an algebraic integer. Let $\beta^{(2)}, \ldots, \beta^{(d)}$ be the conjugates of $\beta = \beta^{(1)}$. As the product of all the conjugates of an algebraic integer is a positive integer, we get

$$\left| \prod_{j=1}^{d} \left(\sum_{i=0}^{N} (a_i - b_i)(\beta^{(j)})^i \right) \right| \geqslant 1.$$

As all conjugates of β have a modulus strictly less than 1 and $|a_i - b_i| \leqslant 2\lfloor \beta \rfloor$,

$$\left| \sum_{i=0}^{N} (a_i - b_i)\beta^i \right| > \frac{1}{\prod_{j=2}^{d} \frac{2\lfloor \beta \rfloor}{1 - |\beta^{(j)}|}}.$$

Since this bound is independent of N, $\mathbb{Z}_\beta$ is uniformly discrete. Using Lemma 1, $\mathbb{Z}_\beta$ is a Delaunay set.

The uniform discreteness of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ can be proved as above with $|a_i - b_i| \leqslant 4\lfloor \beta \rfloor$. Moreover, as $\mathbb{Z}_\beta$ is a Delaunay set, $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ is relatively dense, thus it is a Meyer set. $\square$

3. Automata for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$

In this section, we construct automata that symbolically describe the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ when β is a Parry number. This simple symbolical description of the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ will be used, in the following sections, to determine minimal sets F associated with the Meyer set $\mathbb{Z}_\beta$ when β is a Pisot number.

3.1. Minimal automaton for $\mathbb{Z}_\beta$

When β is a Parry number, the set L_β^+ is recognizable by a minimal finite automaton [13], of which we recall the construction. The reader is referred to [9] and [28] for definitions and results in automata theory. Let us recall the classical construction of the minimal automaton recognizing a language L. The right congruence modulo L is defined as follows: two words v and w are congruent modulo L if they have the same right contexts, more precisely $v \sim_L w$ if $vu \in L$ if and only if $wu \in L$. The minimal automaton of L is then constructed as follows: the states are the right classes mod L, denoted by $[.]_L$. There is a transition from $[v]_L$ to $[v']_L$ labelled by a if $[v']_L = [va]_L$. The initial state is $[\varepsilon]_L$. A state $[v]_L$ is terminal if v belongs to L.

If $d_\beta(1) = t_1 \cdots t_m$ is finite, the automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ recognizing L_β^+ has m states, denoted $0, 1, \ldots, m-1$. The name of state i stands for $[t_1 \cdots t_i]_{L_\beta^+}$, and $0 = [\varepsilon]_{L_\beta^+}$. Denote by suff_k the suffix of $d_\beta^*(1)$ starting at index $k \geqslant 1$. Note that, because of the admissibility condition, the right context of state i is entirely determined by suff_{i+1}, which is the greatest word in the lexicographic order that can be read from i. For each $0 \leqslant i \leqslant m-2$ there is an edge between states i and $i+1$ labelled by t_{i+1}. For each $0 \leqslant i \leqslant m-1$ there are t_{i+1} edges between states i and 0 labelled by $0, 1, \ldots, t_{i+1}-1$. The initial state is 0; every state is terminal. The automaton is shown in Fig. 1.

The case where $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$ is infinite eventually periodic is similar. The automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ recognizing L_β^+ has $m+p$ states $0, \ldots, m+p-1$. For each $0 \leqslant i \leqslant m+p-2$ there is an edge between i and $i+1$ labelled by t_{i+1}. For each $0 \leqslant i \leqslant m+p-1$ there are t_{i+1} edges between i and 0 labelled by $0, \ldots, t_{i+1}-1$. There is an edge from $m+p-1$ to m labelled by t_{m+p}. The initial state is 0; every state is terminal. The automaton is shown in Fig. 2.

We introduce some notations. Set $\bar{k} = -k$, where k is an integer, and let $\overline{A_\beta} = \{\overline{\lfloor \beta \rfloor}, \ldots, \bar{1}, 0\}$. We denote by $L_\beta^- \subset \overline{A_\beta}^*$ the set $\{\overline{w} = \overline{w_N} \cdots \overline{w_0} \mid w = w_N \cdots w_0 = \langle -x \rangle_\beta,\ x \in \mathbb{Z}_\beta^-\}$.

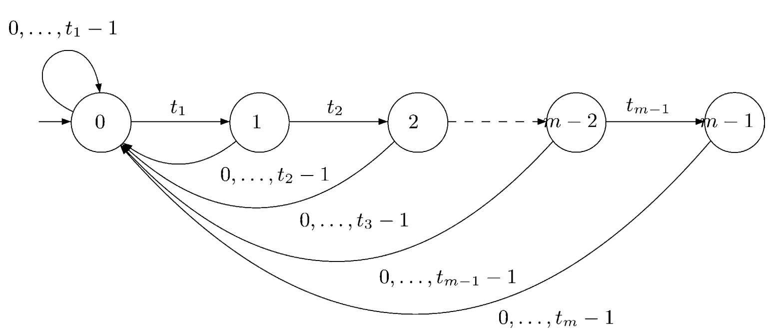

Fig. 1. Automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ when $d_\beta(1) = t_1 \cdots t_m$.

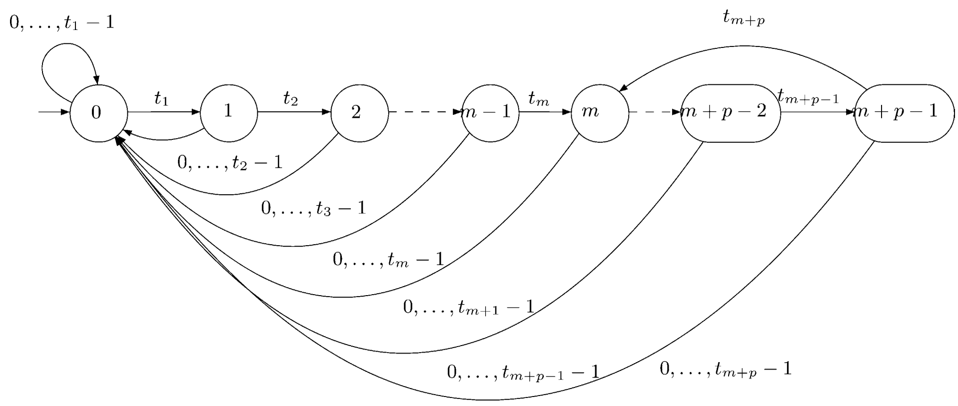

Fig. 2. Automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ when $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+p})^\omega$.

Clearly, the set L_β^- is recognizable by the same automaton as L_β^+, but with negative labels on edges. Then, the set $L_\beta = L_\beta^+ \cup L_\beta^-$ of β-expansions of the elements of $\mathbb{Z}_\beta$ is recognized by the finite automaton $\mathcal{A}_{\mathbb{Z}_\beta} = \mathcal{A}_{\mathbb{Z}_\beta^+} \cup \mathcal{A}_{\mathbb{Z}_\beta^-}$. We will say that the set $\mathbb{Z}_\beta$ is *recognized* by $\mathcal{A}_{\mathbb{Z}_\beta}$.

Example 2. Take $\beta = \frac{1+\sqrt{5}}{2}$. Minimal automata $\mathcal{A}_{\mathbb{Z}_\beta^+}$, $\mathcal{A}_{\mathbb{Z}_\beta^-}$ and $\mathcal{A}_{\mathbb{Z}_\beta}$ are given in Fig. 3.

Since

$$\mathbb{Z}_\beta - \mathbb{Z}_\beta = (\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+) \cup (\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+) \cup -(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+), \tag{1}$$

we introduce symbolic representations of $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+$ and $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$. More precisely, the *formal addition* of elements of $\mathbb{Z}_\beta^+$ consists in adding elements without carry. More precisely,

$$L_\beta^+ + L_\beta^+ = \{(a_N + b_N) \cdots (a_0 + b_0) \mid N \geqslant 0,\ a_N \cdots a_0,\ b_N \cdots b_0 \in L_\beta^+\} \subset \{0, \ldots, 2\lfloor\beta\rfloor\}^*.$$

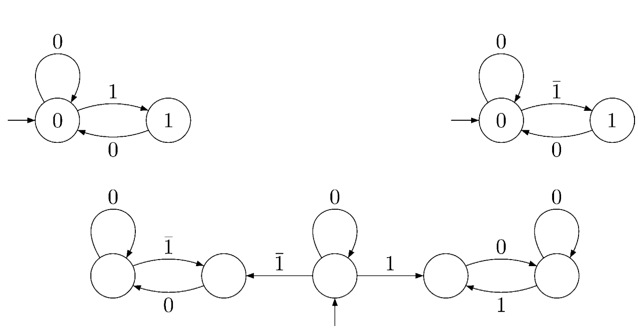

Fig. 3. Automata $\mathcal{A}_{\mathbb{Z}_\beta^+}$, $\mathcal{A}_{\mathbb{Z}_\beta^-}$ and $\mathcal{A}_{\mathbb{Z}_\beta}$.

Similarly, the *formal subtraction* of elements of $\mathbb{Z}_\beta^+$ is defined by

$$L_\beta^+ - L_\beta^+ = \{(a_N - b_N)\cdots(a_0 - b_0) \mid N \geqslant 0,\ a_N \cdots a_0,\ b_N \cdots b_0 \in L_\beta^+\} \subset \{-\lfloor\beta\rfloor, \ldots, \lfloor\beta\rfloor\}^*.$$

3.2. Minimal automaton of $L_\beta^+ + L_\beta^+$

We give a direct construction of the minimal automaton of $L_\beta^+ + L_\beta^+$ when β is a Parry number. Let $Q = \{0, 1, \ldots, h-1\}$ be the set of states of the minimal automaton of L_β^+ ($h = m$ or $h = m + p$ according to the value of $d_\beta(1)$, see Section 3.1).

We construct an automaton $\mathcal{S}$ as follows.
The set of states is the set $Q_\mathcal{S} = \{(i,j) \in Q^2 \mid i \leqslant j\}$. The cardinality of this set is equal to $h(h+1)/2$. The initial state is $(0,0)$ and every state is terminal.
Let c be in $\{0, \ldots, 2\lfloor\beta\rfloor\}^*$, and let (i,j) be in $Q_\mathcal{S}$. Let $\mathcal{C}_c(i,j) = \{(i',j') \in Q^2 \mid \exists a, b \in A_\beta,\ c = a + b,\ i \xrightarrow{a} i' \text{ and } j \xrightarrow{b} j' \text{ in } \mathcal{A}_{\mathbb{Z}_\beta^+}\}$. If $\mathcal{C}_c(i,j)$ is empty there is no transition outgoing from state (i,j) with label c.

Suppose that $\mathcal{C}_c(i,j)$ is not empty. Let $(i',j') \in \mathcal{C}_c(i,j)$. We have seen in Section 3.1 that the right context modulo L_β^+ of state i' is entirely determined by $\mathrm{suff}_{i'+1}$, and similarly for j'. Take $(r,s) \in \mathcal{C}_c(i,j)$ such that $\mathrm{suff}_{r+1} + \mathrm{suff}_{s+1} \geqslant_{\mathrm{lex}} \mathrm{suff}_{i'+1} + \mathrm{suff}_{j'+1}$ for all $(i',j') \in \mathcal{C}_c(i,j)$. This choice ensures that the future readings will be the greatest possible in the lexicographic order. Then, we define in $\mathcal{S}$ a transition $(i,j) \xrightarrow{c} (r,s)$ if $r \leqslant s$, or a transition $(i,j) \xrightarrow{c} (s,r)$ otherwise.

Thus, the following holds true.

Proposition 2. *The automaton $\mathcal{S}$ is the minimal automaton of $L_\beta^+ + L_\beta^+$.*

3.3. Minimal automaton of $L_\beta^+ - L_\beta^+$

We construct an automaton $\mathcal{D}$ for $L_\beta^+ - L_\beta^+$ as follows.

The set of states is the set $Q_\mathcal{D} = \{(i,0), (0,i) \in Q^2 \mid 0 \leqslant i \leqslant h-1\}$. The cardinality of this set is equal to $2h - 1$. The initial state is $(0,0)$ and every state is terminal.

Let c be in $\{0, \ldots, \lfloor\beta\rfloor\}^*$ and let (i,j) be in $Q_\mathcal{D}$. If $c = t_{i+1}$ and if $i \xrightarrow{c} i+1$ in $\mathcal{A}_{\mathbb{Z}_\beta^+}$ we define in $\mathcal{D}$ a transition $(i,j) \xrightarrow{c} (i+1, 0)$. If $c < t_{i+1}$ we define a transition $(i,j) \xrightarrow{c} (0,0)$. Symmetrically, if $\bar{c} = -t_{j+1}$ and if $j \xrightarrow{c} j+1$ in $\mathcal{A}_{\mathbb{Z}_\beta^+}$ we define a transition $(i,j) \xrightarrow{\bar{c}} (0, j+1)$. If $\bar{c} > -t_{j+1}$ there is a transition $(i,j) \xrightarrow{\bar{c}} (0,0)$. In each case the future readings will be the greatest possible in the lexicographic order. Thus, the following holds true.

Proposition 3. *The automaton $\mathcal{D}$ is the minimal automaton of $L_\beta^+ - L_\beta^+$.*

3.4. Fibonacci example

Example 3. In Fig. 4 are drawn the minimal automata $\mathcal{A}_{\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+}$, and $\mathcal{A}_{\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+}$ in the case where $\beta = \frac{1+\sqrt{5}}{2}$. Every state is terminal.

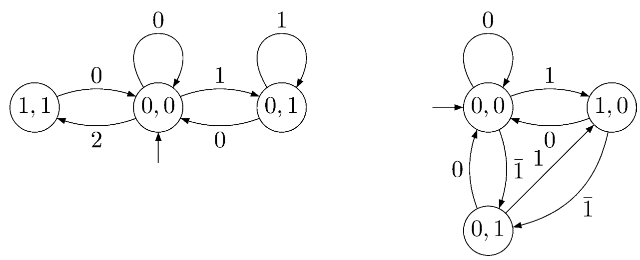

Fig. 4. Automata $\mathcal{A}_{\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+}$ and $\mathcal{A}_{\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+}$.

4. A family of finite sets containing a minimal set F

When β is a Pisot number, the set of β-integers $\mathbb{Z}_\beta$ is a Meyer set so there exists a finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Our goal is to construct sets F as small as possible for $\mathbb{Z}_\beta$.

Note the following property of minimal sets F.

Lemma 2. *If F is a set of minimal size such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ then*

$$F \subset \left(\mathbb{Z}_\beta - \mathbb{Z}_\beta\right) - \mathbb{Z}_\beta.$$

Proof. Let F be a set of minimal size such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$, that is

$$\forall x \in \mathbb{Z}_\beta - \mathbb{Z}_\beta,\ \exists\,(y, f) \in \mathbb{Z}_\beta \times F \ \text{ such that } \ x = y + f.$$

If there exists $f \in F$ such that for all $x \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$ and for all $y \in \mathbb{Z}_\beta$, $f \neq x - y$ then $F' = F \setminus \{f\}$ satisfies $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F'$ and F' is strictly smaller than F, that is contradictory with F minimal. □

Note that there may exist several sets F of minimal size.

Example 4. For $\beta = (1 + \sqrt{5})/2$ the possible minimal sets F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ are the following:

1. $F = \{0, \beta - 1, -\beta + 1\} = \{0, \frac{1}{\beta}, -\frac{1}{\beta}\}$, see [7].
2. $F = \{0, \beta - 2, -\beta + 2\} = \{0, \frac{1}{\beta^2}, -\frac{1}{\beta^2}\} \subset [-\frac{1}{2}, \frac{1}{2}]$, see [12].
3. $F = \{0, \beta - 1, -\beta + 2\} = \{0, \frac{1}{\beta}, \frac{1}{\beta^2}\} \subset [0, 1]$.

Proof. To prove 3., suppose from 1. that for x and y in $\mathbb{Z}_\beta$ there exists z in $\mathbb{Z}_\beta$ such that $x - y = z - \frac{1}{\beta}$. Suppose first z in $\mathbb{Z}_\beta^+$. Denote $\langle z \rangle_\beta = z_k \cdots z_0$ and let z_i be the rightmost non-zero digit. If i is even, then $x - y = z^{(1)} + \frac{1}{\beta^2}$ where $z^{(1)}$ has for β-expansion the word $z_k \cdots z_{i+1}(01)^{i/2}0$, and is thus in $\mathbb{Z}_\beta^+$. If i is odd, then $x - y = z^{(2)}$ where $z^{(2)}$ has for β-expansion $z_k \cdots z_{i+1}(01)^{\lceil i/2 \rceil}$. Now suppose that z belongs to $\mathbb{Z}_\beta^-$. Let $\langle -z \rangle_\beta = u = u_k \cdots u_0$. First suppose that $u_0 = 0$, then write u in the form $u'0(01)^\ell 0$ (if necessary u can be prefixed by two zeroes); then $-(x - y) = -z + \frac{1}{\beta}$ is equal to $v^{(1)} - \frac{1}{\beta^2}$ where

$v^{(1)}$ has for β-expansion the word $u'010^{2\ell}$. If $u_0 = 1$, then u can be written as $u'0(01)^{\ell}$; then $-(x-y)$ has for β-expansion the word $u'010^{2\ell-1}$. $\square$

Using properties of the algebraic conjugates of the elements of minimal sets F, we first define finite sets from which can be extracted the finite sets F.

Lemma 3. *Let β be a Pisot number of degree d, let $I \subset \mathbb{R}$ be an interval of finite length greater than or equal to* 1 *and let W be the following set*

$$W = \left\{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and for } 2 \leqslant j \leqslant d,\ |x^{(j)}| < \frac{3\lfloor\beta\rfloor}{1-|\beta^{(j)}|} \right\},$$

where $x^{(2)}, \ldots, x^{(d)}$ are the algebraic conjugates of x. Then W is finite, and $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + W$.

Proof. From Lemma 1 the maximal distance between two consecutive points of $\mathbb{Z}_\beta$ is equal to 1, thus one can find a finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ in any interval I of length greater than or equal to 1. Fix an interval I of length $\geqslant 1$ and let F be a finite subset of I of minimal size such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Let $x \in F$, then from Lemma 2, $x \in (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$ and can be written as

$$x = \sum_{i=0}^{N} (a_i - b_i)\beta^i - \sum_{i=0}^{N} c_i\beta^i \quad \text{with } |a_i|, |b_i|, |c_i| \leqslant \lfloor\beta\rfloor.$$

So

$$\text{for } 2 \leqslant j \leqslant d \quad x^{(j)} = \sum_{i=0}^{N} (a_i - b_i - c_i)(\beta^{(j)})^i \quad \text{with } |a_i - b_i - c_i| \leqslant 3\lfloor\beta\rfloor.$$

As β is a Pisot number, for all $j \geqslant 2$, $|\beta^{(j)}| < 1$ and $|\sum_{i=0}^{N}(\beta^{(j)})^i| < (1-|\beta^{(j)}|)^{-1}$. We obtain in this way the announced bound on the moduli of the conjugates of x and $x \in W$. So F is a subset of W.

Since β is a Pisot number the set W contains only points of $\mathbb{Z}[\beta]$ with bounded modulus and whose all conjugates have bounded modulus, thus W is finite. $\square$

The choice of any interval $I \subset]-1,1[$ of length 1 allows us to reduce the size of the set containing a minimal set F.

Lemma 4. *Let β be a Pisot number of degree d, let $I \subset]-1,1[$ be an interval of length* 1 *and let U be the following set*

$$U = \left\{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and for } 2 \leqslant j \leqslant d,\ |x^{(j)}| < \frac{2\lfloor\beta\rfloor}{1-|\beta^{(j)}|} \right\}.$$

Then U is finite and $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + U$.

Proof. We choose here $I \subset]-1,1[$ of length 1 and improve the bound on the moduli of the conjugates of x given in Lemma 3 by considering the decomposition

$$\mathbb{Z}_\beta - \mathbb{Z}_\beta = (\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+) \cup (\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+) \cup -(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+).$$

More precisely, let F be a finite subset of I of minimal size such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ and let $x \in F$, then $x \in (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$ and can be written as

$$x = \sum_{i=0}^{N} (a_i - b_i)\beta^i - \sum_{i=0}^{N} c_i \beta^i.$$

We study $|a_i - b_i - c_i|$ according to the signs of a_i, b_i and c_i. Recall that $|a_i|, |b_i|$ and $|c_i|$ are smaller than $\lfloor \beta \rfloor$. In $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$ and $\mathbb{Z}_\beta^- - \mathbb{Z}_\beta^-$, the products $a_i b_i$ are non-negative and the coefficients satisfy $|a_i - b_i| \leqslant \lfloor \beta \rfloor$. When $F \subset]-1,1[$, $\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+ + F$ and $-\left(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+\right) \subset \mathbb{Z}_\beta^- + F$, so when $a_i b_i \leqslant 0$, then $a_i c_i \geqslant 0$ and we have $|a_i - c_i| \leqslant \lfloor \beta \rfloor$. Thus, when $F \subset]-1,1[$, we get in all cases $|a_i - b_i - c_i| \leqslant 2\lfloor \beta \rfloor$. Thus,

$$\text{for } 2 \leqslant j \leqslant d \quad x^{(j)} = \sum_{i=0}^{N} (a_i - b_i - c_i)(\beta^{(j)})^i \ \text{ with } |a_i - b_i - c_i| \leqslant 2\lfloor \beta \rfloor,$$

and the announced bound on the moduli of the conjugates of x holds true. The proof that U is finite is the same as for W. $\square$

Remark 1. In what follows we restrict our study to the sets U defined in Lemma 4 as finite subsets of intervals $I \subset]-1,1[$ of length 1, but all constructions remain valid with small changes for the finite sets W introduced in Lemma 3 as finite subsets of arbitrary intervals of length greater or equal to 1.

4.1. Quadratic Pisot numbers

We now establish a bound on the size of the sets U of Lemma 4 for any quadratic Pisot number β. Recall [13] that a quadratic Pisot number β has a minimal polynomial of the form $M_\beta = X^2 - aX - b$, with either $a \geqslant b \geqslant 1$, or $a \geqslant 3$ and $0 > b \geqslant -a + 2$. In the first case $d_\beta(1) = ab$, and in the second one $d_\beta(1) = (a-1)(a+b-1)^\omega$.

Proposition 4. *Let β be a quadratic Pisot number with minimal polynomial $M_\beta = X^2 - aX - b$. Then for any interval $I \subset]-1,1[$ of length 1, $Card(U) \leqslant 2\lceil B - 1 \rceil + 1$, with*

$$B = \begin{cases} \frac{a}{a-b+1} + \frac{a(a+2)}{(a+1)(a-b+1)} + \frac{1}{a+1} & \textit{when} \quad a \geqslant b > \frac{a}{2}, \\ \frac{2(a+1)}{a-b+1} + \frac{1}{a} & \textit{when} \quad 0 < b \leqslant \frac{a}{2}, \\ \frac{2a-3}{a+b-1} + \frac{1}{a-1} & \textit{when} \quad -\frac{a}{2} < b < 0, \\ \frac{2(a-1)}{a+b-1} + \frac{1}{a-2} & \textit{when} \quad -a+2 \leqslant b \leqslant -\frac{a}{2}. \end{cases}$$

Proof. Denote by β' the algebraic conjugate of β. Any point x of $\mathbb{Z}[\beta]$ and its algebraic conjugate x' can be written as $x = x_1 + x_2\beta$ and $x' = x_1 + x_2\beta'$ where $x_1, x_2 \in \mathbb{Z}$. Then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\beta - \beta'} \begin{pmatrix} -\beta' & \beta \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}.$$

Note that for each value of x_2 there is only one possible value for x_1 such that $x \in U$ since x_1 is an integer and the interval I is of length 1. So if, for all $x \in U$, $|x_2| < B$ then $|x_2| \leqslant \lceil B - 1 \rceil$ and $\mathrm{Card}(U) \leqslant 2\lceil B - 1 \rceil + 1$.

We establish the bound on the modulus of x_2 using the inequalities $|x| < 1$ and $|x'| \leqslant 2\lfloor\beta\rfloor/(1 - |\beta'|)$ with $\lfloor\beta\rfloor = a$ when $b > 0$ and $\lfloor\beta\rfloor = a - 1$ when $b < 0$. Setting $\Delta = a^2 + 4b$, we get when $b > 0$,

$$|x_2| < \frac{1}{\sqrt{\Delta}}\left(1 + \frac{4a(a+2+\sqrt{\Delta})}{(a+2)^2 + \Delta}\right) \leqslant \frac{a}{a-b+1} + \frac{a(a+2)}{\sqrt{\Delta}(a-b+1)} + \frac{1}{\sqrt{\Delta}}$$

and when $b < 0$,

$$|x_2| < \frac{1}{\sqrt{\Delta}}\left(1 + \frac{4(a-1)(a+2+\sqrt{\Delta})}{\Delta - (a-2)^2}\right) \leqslant \frac{a-1}{a+b-1} + \frac{(a-1)(a-2)}{\sqrt{\Delta}(a+b-1)} + \frac{1}{\sqrt{\Delta}}.$$

The announced bounds follow from the study of Δ according to the value of b. □

Remark 2. Specifying the values for a and b given above for B, we obtain the following bounds:

- If $a \geqslant b > \frac{a}{2}$, then $B \leqslant 2a + 1$ and $\mathrm{Card}(U) \leqslant 4a + 1$.
- If $0 < b \leqslant \frac{a}{2}$, then $B < 4$ and $\mathrm{Card}(U) \leqslant 7$.
- If $-\frac{a}{2} < b < 0$, $B < 7$ and $\mathrm{Card}(U) \leqslant 13$.
- If $-a + 2 \leqslant b \leqslant -\frac{a}{2}$ then $B \leqslant 2a - 1$ and $\mathrm{Card}(U) \leqslant 4a - 3$.

Corollary 1. *Let β be a quadratic Pisot unit, i.e., $|b| = 1$, and $I \subset]-1, 1[$ be an interval of length* 1, *then the set U contains at most* 5 *points.*

Proof. From Proposition 4, when $b = 1$ or $b = -1$, $B \leqslant 3$, in all but two cases.

If $M_\beta = X^2 - 3X + 1$, then $B \leqslant 4$ and $|x_2| \leqslant 3$ but there is no corresponding value for x_1 when $|x_2| = 3$, thus $|x_2| \leqslant 2$ and $\mathrm{Card}(U) \leqslant 5$.

If $M_\beta = X^2 - 2X - 1$, we obtain $B \leqslant 3$ if we do not approximate Δ in the computation of the proof of Proposition 4. □

Example 5. Let $\beta = (1 + \sqrt{5})/2$ then $\beta' = (1 - \sqrt{5})/2$. Then

$$U = \left\{x \in \mathbb{Z}[\beta] \mid x \in I \text{ and } |x'| < 2\beta + 2\right\}.$$

- For $I = [-1/2, 1/2[$, $U = \{0, \beta - 2, 2\beta - 3, 2 - \beta, 3 - 2\beta\}$.
- For $I = [0, 1[$, $U = \{0, -1 + \beta, -3 + 2\beta, 2 - \beta\}$, since the conjugate $4 - 2\beta'$ of $4 - 2\beta$ has a modulus greater than $2\beta + 2$.

Example 4 shows that the size of minimal sets F in this case is equal to 3.

5. A reduction of the sets containing minimal sets F

We present our constructions in the case where I is an interval of length 1 in $]-1,1[$ and consider the finite subset U of I defined in Lemma 4. By construction a minimal set F is contained in U and from Lemma 2 F is a subset of $(\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$. Thus, a minimal set F is included in $U \cap ((\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta)$.

In the following, we give an algorithm that computes this intersection. Roughly speaking we construct an automaton that recognizes the Cartesian product $(L_\beta - L_\beta) \times L_\beta$ and whose each state q corresponds to the value of the subtraction of the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ and $\mathbb{Z}_\beta$ whose representations label the paths from the initial state to q.

The first step of the construction consists in associating to each element of a minimal set F at least a path labelled on $\{-2\lfloor\beta\rfloor, \ldots, 2\lfloor\beta\rfloor\}^* \times \{0, \ldots, \lfloor\beta\rfloor\}^*$ in a directed graph G whose set of vertices contains U.

Following [15], we define the directed graph G as follows.

- The set of vertices is

$$V = \left\{ x \in \mathbb{Z}[\beta] \;\middle|\; |x| < \frac{2\lfloor\beta\rfloor}{\beta - 1}, \text{ and for } 2 \leqslant j \leqslant d,\ |x^{(j)}| < \frac{2\lfloor\beta\rfloor}{1 - |\beta^{(j)}|} \right\}.$$

- The labels (b, a) of the transitions belong to $\{-2\lfloor\beta\rfloor, \ldots, 2\lfloor\beta\rfloor\} \times \{0, \ldots, \lfloor\beta\rfloor\}$.
- There is a transition from $x \in V$ to $y \in V$ labelled by (b, a), denoted $x \xrightarrow{(b,a)} y$, if and only if $y = \beta x + (b - a)$.

Note that $0 \in V$ and $U \subset V$. The set V is finite.

Remark 3. Transitions in G are defined in such a way that words will be processed most significant digit first (*i.e.*, from left to right) as in the automata for $\mathbb{Z}_\beta$ and $\mathbb{Z}_\beta - \mathbb{Z}_\beta$.

Proposition 5. *Let $F \subset U$ be a minimal set satisfying $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Then for any $f \in F$ there is a path from* 0 *to f whose label belongs to $(L_\beta - L_\beta) \times L_\beta$.*

Proof. From Lemma 2, $F \subset (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$, so any element f of F can be written as $f = \sum_{i=0}^{N} (b_i - a_i)\beta^i$ where $x = \sum_{i=0}^{N} a_i\beta^i \in \mathbb{Z}_\beta$ with $a_N \cdots a_0 \in L_\beta$ and $y = \sum_{i=0}^{N} b_i\beta^i \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$ with $b_N \cdots b_0 \in L_\beta - L_\beta$.

With such an f is associated a finite sequence

$$f_0 = 0, \quad \text{for } 0 \leqslant i \leqslant N \quad f_{i+1} = \beta f_i + (b_{N-i} - a_{N-i}).$$

Note that $f_{N+1} = f$.

Let us show that for any $f \in F$, the elements $f_1, \ldots, f_{N+1}$ of the sequence associated with f belong to V. Note that the smallest K such that $|x| < K$ implies $|(x - (b - a))/\beta| < K$ is $K = 2\lfloor\beta\rfloor/(\beta - 1)$. Since f is in U, $|f| < K$, and so for all $0 \leqslant i \leqslant N$, $|f_i| < K$. Moreover from Lemma 4, when $F \subset U$, for all i, $|b_i - a_i| \leqslant 2\lfloor\beta\rfloor$, thus for $1 \leqslant i \leqslant N+1$ and $2 \leqslant j \leqslant d$, the conjugates $f_i^{(j)}$ of f_i satisfy $|f_i^{(j)}| \leqslant 2\lfloor\beta\rfloor/(1 - |\beta^{(j)}|)$ and for $1 \leqslant i \leqslant N+1$, f_i belongs to V.

Finally, if $f \in F$ then there is in G a path

$$0 = f_0 \overset{(b_N,a_N)}{\longrightarrow} f_1 \overset{(b_{N-1},a_{N-1})}{\longrightarrow} \cdots \overset{(b_0,a_0)}{\longrightarrow} f_{N+1} = f,$$

where the words $a_N \cdots a_0$ and $b_N \cdots b_0$, respectively, belong to L_β and $L_\beta - L_\beta$, concluding the proof. □

From Proposition 5, we can take into account in G only the paths whose labels belong to $(L_\beta - L_\beta) \times L_\beta$. In order to compute such paths, we use the Cartesian product of the automata $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}$ and $\mathcal{A}_{\mathbb{Z}_\beta}$. Recall the definition of the Cartesian product $\mathcal{P} = \mathcal{A} \times \mathcal{B}$ of two automata $\mathcal{A}$ and $\mathcal{B}$:

- the set of states of $\mathcal{P}$ is $Q_\mathcal{P} = Q_\mathcal{A} \times Q_\mathcal{B}$,
- there is an edge in $\mathcal{P}$ from (p,q) to (p',q') labelled by (a,b) if and only if there is an edge from p to p' labelled by a in $\mathcal{A}$ and an edge from q to q' labelled by b in $\mathcal{B}$,
- the set of initial (resp. terminal) states of $\mathcal{P}$ is the Cartesian product of the sets of initial (resp. terminal) states of $\mathcal{A}$ and $\mathcal{B}$.

Note that in $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}$ every state is terminal.

From all vertices f of G which are in U we look for a path from 0 to f in the directed graph G which is successful in $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}$. We find these paths making use of the intersection $\mathcal{I} = \mathcal{A} \cap \mathcal{B}$ of two finite automata $\mathcal{A}$ and $\mathcal{B}$ defined as follows:

- all sets of states of $\mathcal{I}$ are defined as the ones of the Cartesian product,
- there is an edge in $\mathcal{I}$ from (p,q) to (p',q') labelled by a if and only if there is an edge from p to p' in $\mathcal{A}$ and an edge from q to q' in $\mathcal{B}$ both labelled by a.

Algorithm of reduction of the size of the sets containing a minimal set *F*
Input: The set U containing a minimal set F.
Output: A subset U' of U containing a minimal set F.

1. Build the automaton $\mathcal{G}_U$ having as underlying transition graph G with 0 as initial state and U as set of terminal states.
2. Compute the intersection $\mathcal{I}_U = (\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}) \cap \mathcal{G}_U$. Note that the set of terminal states of $\mathcal{I}_U$ is $Q_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times Q_{\mathbb{Z}_\beta} \times U$.
3. Prune $\mathcal{I}_U$ into $\mathcal{I}'_{U'}$ (that is, keep only the states which belong to a path from the initial state to a terminal state).
4. Return the set U' of the third components of terminal states of $\mathcal{I}'_{U'}$.

Corollary 2. *A minimal set F is contained in $U' \subset U$.*

Remark 4. The number of states of the automaton $\mathcal{I}_{U'}$ is $\mathcal{O}\left(Q^3 \times |V|\right)$, where Q is the number of states of $\mathcal{A}_{\mathbb{Z}^+_\beta}$ and $|V|$ is the number of vertices of G.

Because of the large number of states of the automaton obtained in this way, we shall not illustrate the construction with a figure. Nevertheless, we give an example of reductions that can be obtained.

Example 6. When $\beta = (1+\sqrt{5})/2$, we obtain

- For $I = [-1/2, 1/2[$ and $U = \{0, \beta - 2, 2\beta - 3, 2 - \beta, 3 - 2\beta\}$,

$$U \cap (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta = \{0, \beta - 2, 2 - \beta\}.$$

- For $I = [0, 1[$ and $U = \{0, -1 + \beta, -3 + 2\beta, 2 - \beta\}$,

$$U \cap (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta = \{0, \beta - 1, 2 - \beta\}.$$

A geometrical argument could also be used to prove that $2\beta - 3 = \frac{1}{\beta^3}$ and $-2\beta + 3 = -\frac{1}{\beta^3}$ are not in $(\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$. Indeed, the distance between two consecutive points of $\mathbb{Z}_\beta$ is equal to $\frac{1}{\beta}$ or $1 = \frac{1}{\beta} + \frac{1}{\beta^2}$, so $\mathbb{Z}_\beta + \{\frac{1}{\beta^3}, -\frac{1}{\beta^3}\} \cap \mathbb{Z}_\beta + \{0, \frac{1}{\beta}, -\frac{1}{\beta}\} = \emptyset$. Moreover, $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \frac{1}{\beta}, -\frac{1}{\beta}\}$ (see Example 4), thus $\mathbb{Z}_\beta - \mathbb{Z}_\beta \cap \mathbb{Z}_\beta + \{\frac{1}{\beta^3}, -\frac{1}{\beta^3}\} = \emptyset$ and $\pm\frac{1}{\beta^3} \notin (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$.

6. Algorithm computing a minimal set *F*

The finite sets U' obtained by the previous construction are not minimal. An element $y \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$ can be close to two distinct points of x and x' of $\mathbb{Z}_\beta$, for example, such that $x < y < x'$, and $y = x + f = x' + f'$ with $f, f' \in U'$.

Theorem 1. *A minimal set $F \subset U'$ can be computed by an algorithm which is exponential in time and space. It consists in building a transducer which rewrites a representation of an element of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ into its representation $\mathbb{Z}_\beta + F$.*

Proof. To find a minimal set $F \subset U'$ we proceed in two steps.

First, we define from the automaton $\mathcal{I}'_{U'}$ a deterministic automaton $\mathcal{R}_{U'}$ that recognizes the set $L_\beta - L_\beta$. Note that the words of $L_\beta - L_\beta$ appear as the first component of the labels of the successful paths in $\mathcal{I}'_{U'}$. The automaton $\mathcal{R}_{U'}$ is obtained by erasing the second component of the labels (that belongs to L_β) of the transitions of $\mathcal{I}'_{U'}$ and determining the automaton defined in this way. The determination of automata is based on the so-called subset construction (see [9]), which is exponential in space, and the automaton $\mathcal{R}_{U'}$ has $\mathcal{O}(2^{Q_{\mathcal{I}'_{U'}}})$ states.

Next, we look amongst all subsets of U' for the smallest set F such that the automaton $\mathcal{R}_F$, obtained from $\mathcal{R}_{U'}$ by keeping only as terminal states the terminal states of $\mathcal{R}_{U'}$ in which occur an element of F, recognizes $L_\beta - L_\beta$. To test the inclusion, we compute the complement $\mathcal{C}_F$ of $\mathcal{R}_F$ by completing the automaton $\mathcal{R}_F$ (when a transition is missing we add a transition ending in a new state called the sink) and replacing the set of terminal states F by its complement (including the sink). Then the automaton $\mathcal{R}_F$ recognizes $L_\beta - L_\beta$ if and only if the intersection of $\mathcal{C}_F$ and $\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}$ is empty. Note that the complexity of the search amongst all subsets of U' is exponential in time.

From the set F obtained above, we define a transducer that provides, for any $b = b_N \cdots b_0 \in L_\beta - L_\beta$ and $y = \sum_{i=0}^{N} b_i \beta^i \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$, a decomposition $(a_N \cdots a_0, f)$ where $a = a_N \cdots a_0 \in L_\beta$, $f \in F$ and $y = \sum_{i=0}^{N} a_i \beta^i + f$.

Consider $\mathcal{I}_F = (\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times \mathcal{A}_{\mathbb{Z}_\beta}) \cap \mathcal{G}_F$ (F is the set of terminal states of $\mathcal{G}_F$). For any element $b = b_N \cdots b_0 \in L_\beta - L_\beta$ there exists $f \in F$ such that b is the first component of the label of a successful path w ending in (s, f) where s is any state of $(\mathcal{A}_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}) \times \mathcal{A}_{\mathbb{Z}_\beta}$ (by construction all states are terminal). Consequently, we get $\sum_{i=0}^{N} b_i \beta^i = \sum_{i=0}^{N} a_i + f$ where $a_N \cdots a_0$ is the second component of the label of the same path w and so belongs to L_β.

More generally the first component of the labels of the edges in $\mathcal{I}_F$ can be interpreted as the inputs in $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ given by their representation in $L_\beta - L_\beta$ of the transducer, the second component as the corresponding outputs in $\mathbb{Z}_\beta$ given by their representation in L_β. The associated element of F is given by the second component of the label of the state where the path ends. $\square$

To conclude, the method used here for determining minimal sets F probably could be generalized to the following sets. Let G be a strongly connected graph labelled by numbers taken from a finite alphabet, and let β be the spectral radius of its adjacency matrix. Let us consider the set $X_G = \{\sum_{i=0}^{k} x_i \beta^i \mid k \geqslant 0,\ x_k \cdots x_0$ is the label of a path in $G\}$. Under certain conditions on G and β, X_G is a Meyer set, and so the question of minimal F makes sense. The characterization of these Meyer sets and the construction of associated minimal sets F remain open problems.

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